Motivic classes of varieties and stacks with applications to Higgs bundles

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Overview

Introduction

Our primary goal is to give an introduction to motivic classes of varieties and stacks with applications to moduli of Higgs bundles.

Sources of Motivation:

- The work of Hausel, Letellier, Rodriguez-Villegas, and others regarding mixed Hodge polynomials of character varieties.
- Point counting for algebraic varieties and stacks over a finite field $\mathbb{F}_q$. Specifically, computations for moduli stacks of Higgs bundles done by Mozgovoy, Schiffmann, and Mellit.

Counting points of varieties over a finite field

Example

Let $|X| := |X(\mathbb{F}_q)|$ denote the number of rational points of an algebraic variety X over a finite field $\mathbb{F}_q$ with q elements.

- $|\mathbb{A}^n| = q^n$.
- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathrm{GL}(n)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$ considered as the space of n linearly independent columns.

Motivic classes of varieties

Preliminary definition

For any field k, one defines the abelian group $\mathrm{Mot}_{var}(k)$ as the group generated by isomorphism classes of varieties over k modulo the following relations:

$$[X] = [Y] + [X - Y] \text{ where } Y \text{ is a closed subvariety of } X.$$

The class $[X]$ in $\mathrm{Mot}_{var}(k)$ is called the motivic class of the variety X. We can give a ring structure for $\mathrm{Mot}_{var}(k)$ by $[X] \cdot [Y] := [(X \times_k Y)_{red}]$ with the unit element $[\mathrm{Spec}\, k] = 1$.

Note that $[\emptyset] = 0$. Similarly we can define the motivic classes of schemes of finite type over k,by $[X] := [X_{red}]$.

Realizations of the motivic classes

Lemma

Let A be an abelian group and f be an A-valued function on isomorphism classes of k-varieties such that for all closed subvarieties $Y \subset X$, we have $f(X) = f(Y) + f(X - Y)$. Then there is a unique $\tilde{f} : \mathrm{Mot}_{var}(k) \to A$ such that for all varieties X we have $\tilde{f}([X]) = f(X)$.

Example

- Point counting: For $k = \mathbb{F}_q$, define $\# : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$ as $\#([X]) = |X(\mathbb{F}_q)|$, the number of rational points over $\mathbb{F}_q$.
- Euler characteristic: For $k = \mathbb{C}$, define $\chi : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$ such that $\chi([X]) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q})$ is the Euler characteristic of X.
- E-polynomial: For $k = \mathbb{C}$, define $E : \mathrm{Mot}_{var}(k) \to \mathbb{Z}[u, v]$ by setting $E([X]) = \sum_{k,p,q} (-1)^k h^{k,p,q} u^p v^q$ such that $h^{k,p,q} = \dim_{\mathbb{C}} H_c^{k,p,q}(X)$ for any quasi-projective variety X.

Examples

So, instead of performing computations of invariants directly, we can instead perform computations of motivic classes in $\mathrm{Mot}_{var}(k)$. Letting $[\mathbb{A}^1_k] = \mathbb{L}$, we see that the same methods used for counting points over $\mathbb{F}_q$ give us:

Example

- $[\mathbb{A}^n] = \mathbb{L}^n$.
- $[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n$.
- $[\mathrm{GL}(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$.

Local zeta function

Definition

The local zeta function of a variety X is defined as

$$Z(X,t) = \exp\left(\sum_{k=1}^{\infty} \frac{|X(\mathbb{F}_{q^k})|}{k} t^k\right)$$

We prefer to rewrite local zeta function in terms of symmetric products.

Proposition

$$Z(X,t) = \sum_{n\geq 0} |Sym^n X(\mathbb{F}_q)| \cdot t^n$$

where $Sym^n X$ is the symmetric product X^n/S_n (with the convention that $Sym^0 X = \operatorname{Spec} k$).

Compare the two versions of local zeta function

Example

For the original local zeta function, the coefficient at t^2 is:

$$\frac{|X(\mathbb{F}_{q^2})|}{2} + \frac{(|X(\mathbb{F}_q)|)^2}{2}.$$

For the symmetric product $|Sym^2X(\mathbb{F}_q)|$, we have the following two cases. The contribution from pairs (x, y), where $x, y \in \mathbb{F}_q$ is $|Sym^2(X(\mathbb{F}_q))|$. If $x \in \mathbb{F}_{q^2} - \mathbb{F}_q$, then $(x, \phi(x))$ (where ϕ is a Frobenius morphism) is in $Sym^2X(\mathbb{F}_q)$. The contribution of these points is,

$$\frac{|X(\mathbb{F}_{q^2})| - |X(\mathbb{F}_q)|}{2}.$$

Finally we have

$$|Sym^2(X(\mathbb{F}_q))| + \frac{|X(\mathbb{F}_{q^2})| - |X(\mathbb{F}_q)|}{2} = \frac{|X(\mathbb{F}_{q^2})|}{2} + \frac{(|X(\mathbb{F}_q)|)^2}{2}.$$

Motivic zeta function

Definition

We can define the motivic zeta function for every quasi-projective variety X:

$$Z_X(t) = \sum_{n \geq 0} [Sym^n X] \cdot t^n \in 1 + t \cdot \mathrm{Mot}_{var}(k)[[t]].$$

- If $k = \mathbb{F}_q$, then the image of the series $Z_X(t)$ under the counting measure

$$\# : \mathrm{Mot}_{var}(k) \to \mathbb{Z} \text{ given by } [X] \mapsto |X(\mathbb{F}_q)|$$

coincides with the local zeta function.

Motivic classes of varieties

Definition

For any field k, one defines the abelian group $\mathrm{Mot}_{var}(k)$ as the group generated by isomorphism classes of varieties over k modulo the following relations:
1) $[X] = [Y] + [X - Y]$ where Y is a closed subvariety of X,
2) $[X] = [Y]$ if there is a surjective and universally injective morphism $X \to Y$ of varieties over k.
The class $[X]$ in $\mathrm{Mot}_{var}(k)$ is called the motivic class of the variety X. We can give a ring structure for $\mathrm{Mot}_{var}(k)$ by $[X] \cdot [Y] := [(X \times_k Y)_{red}]$ with the unit element $[\mathrm{Spec}\, k] = 1$.

Note that $[\emptyset] = 0$. Similarly we can define the motivic classes of schemes of finite type over k,by $[X] := [X_{red}]$.

Remarks

- In characteristic 0, $\mathrm{Mot}_{var}(k)$ coincides with the the first definition. In fact, the second relation is always true because every surjective and universally injective morphism becomes an isomorphism after passing to locally closed stratifications.

Surjective and universally injective relation

We want to extend the motivic zeta function to $\mathrm{Mot}(k)$. For this we need to relate $Sym^n X$ to $Sym^n Z$ and $Sym^n(X - Z)$.

Lemma

Let X be a quasiprojective variety over k, and G an abstract finite group acting on X by algebraic automorphisms over k. Let $\pi : X \to X/G$ be the quotient morphism. Let Z be a G-invariant closed subvariety of X. We have in $\mathrm{Mot}_{var}(k)$

$$[X/G] = [Z/G] + [(X - Z)/G].$$

Proof of Lemma

Let $\pi : X \to X/G$ be the projection. The topology on X/G is the usual quotient topology. Then $\pi(Z)$ is closed, so we have

$$[X/G] = [\pi(Z)] + [\pi(X - Z)].$$

In finite characteristic, the image of Z in X/G is not in general isomorphic to Z/G. However, there is a natural surjective and universally injective morphism $Z/G \to \pi(Z)$. Thus we have

$$[Z/G] = [\pi(Z)].$$

On the other hand, since $X - Z$ is an open subvariety, then $\pi(X - Z)$ is open in X/G with the quotient topology. Now we have the following isomorphism

$$(X - Z)/G \simeq \pi(X - Z).$$

Properties of motivic zeta function

Proposition (Totaro for b))

If X is a quasi-projective variety, Y is a closed subvariety of X, then
a) $Z_X(t) = Z_Y(t)Z_{X-Y}(t)$,
b) $Z_{X\times\mathbb{A}_k^n}(t) = Z_X(\mathbb{L}^n t)$.

Example

If X is a quasi-projective variety, then for every vector bundle $E \to X$ of dimension n, we have

$$Z_E(t) = Z_X(\mathbb{L}^n t).$$

Counting points of projective spaces

We are interested in performing computations for quotients. One of the motivation comes from the following computations for projective spaces.

Example

- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathbb{P}^n|$ can also be computed by presenting $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1} - \{0\}$ by the action of $\mathrm{GL}(1)$:

$$\left|\frac{\mathbb{A}^{n+1} - \{0\}}{\mathrm{GL}(1)}\right| = \frac{q^{n+1} - 1}{q - 1} = 1 + q + \cdots + q^n.$$

Counting points of algebraic stacks over a finite field

- We can try to do similar computations even when the group action is not free and the quotient is no longer a variety.
- Algebraic stacks generalize varieties by allowing points to have nontrivial automorphisms.
- Examples include quotients X/G of a variety by a non-free algebraic group action, moduli of curves, moduli of vector bundles, etc.

If $\mathcal{Y}$ is an algebraic stack over $\mathbb{F}_q$, we can generalize point counting for varieties by defining the volume (also known as the mass) of $\mathcal{Y}$ as a weighted sum over all its $\mathbb{F}_q$-rational points:

$$|\mathcal{Y}| := \sum_{y \in \mathrm{Ob}(\mathcal{Y}(\mathbb{F}_q))} \frac{1}{|\operatorname{Aut}(y)|}.$$

This is not guaranteed to converge. However, most of the stacks we consider are of finite type, for which it converges (the sum is finite).

Counting points of groupoids

If $\mathcal{G}$ is a groupoid, we can define the volume of $\mathcal{G}$ as a weighted sum:

$$|\mathcal{G}| := \sum_{G \in \mathcal{G}} \frac{1}{|\operatorname{Aut}(G)|}.$$

Example (Funny)

The volume of the groupoid of finite sets is e. Indeed, for any finite set of size n, the automorphism group is of order $n!$:

$$|\mathcal{G}| = \sum_{G \in \mathcal{G}} \frac{1}{|\operatorname{Aut}(G)|} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Example: volume of the stack of vector bundles

We give an example where the stack is not of finite type but the volume converges.

Siegel formula

Let X be a smooth projective curve of genus g over $\mathbb{F}_q$. Let $\mathcal{B}un_{r,d}(X)$ be the moduli stack parametrizing isomorphism classes of rank r, degree d vector bundles over X. The volume over $\mathbb{F}_q$ is

$$|\mathcal{B}un_{r,d}(X)| = \frac{q^{(r^2-1)(g-1)}}{q-1}|Jac(X)|\prod_{i=2}^{r} Z_X(q^{-i})$$

where $Jac(X)$ is the Jacobian of X, i.e. the moduli space of degree 0 line bundles on X
(the volume of $|Jac(X)|$ is also easily computed using the zeta function; it is closely related to $res_{t=1}Z_X(t)$).

Motivic classes of stacks

We use the convention that all the stacks considered will be Artin stacks locally of finite type over a field k such that the stablizers of points are affine.

Definition

One defines the abelian group $\mathrm{Mot}(k)$ as the group generated by isomorphism classes of stacks of finite type over k modulo the following relations:

1) $[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} - \mathcal{Y}]$ where $\mathcal{Y}$ is a closed substack of $\mathcal{X}$,

2) $[\mathcal{X}] = [\mathcal{Y}]$ if there is a surjective and universally injective morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over k.

3) $[\mathcal{X}] = [\mathcal{Y} \times \mathbb{A}^r_k]$ where $\mathcal{X} \to \mathcal{Y}$ is a vector bundle of rank r.

The class $[\mathcal{X}]$ in $\mathrm{Mot}(k)$ is called the motivic class of the stack $\mathcal{X}$.

Similarly we can give a ring structure for $\mathrm{Mot}(k)$ by $[\mathcal{X}] \cdot [\mathcal{Y}] := [\mathcal{X} \times_k \mathcal{Y}]$.

Remarks

- For the motivic classes of varieties, the third relation is trivial since every vector bundle over a variety is locally trivial in the Zariski topology. However in motivic classes of stacks, we have a vector bundle
$$V := \mathbb{A}_k^n \times^{\mathrm{GL}(n)} \{pt\} \to \mathrm{BGL}(n) := */\mathrm{GL}(n),$$
but it is not Zariski locally trivial.

Quotient stacks

Definition

Let X be a scheme over k and let G be a linear algebraic group acting on X. We define the quotient stack X/G as

$$X/G : (Sch/k)^{op} \to Grpds$$

$$S \mapsto \{(E, \alpha) : E \xrightarrow{G} S \text{ is a } G\text{-torsor and}$$
$$\alpha : E \to X \text{ is a } G\text{-equivariant morphism}\}$$
$$+ \{\text{isomorphism of pairs}\}$$

Example

When $X = *$, we have the classifying stack $BG := */G$. It classifies all the G-torsors.

Affine stabilizer condition

We need the affine stabilizer condition since every such stack has a stratification by global quotients of the form $X/\operatorname{GL}(n)$ with X a variety.

Proposition (Kresch, 1999)

(1) Let $\mathcal{X}$ be a stack. Then $\mathcal{X}$ admits a stratification by quotient stacks if and only if for every geometric point the stabilizer group is affine.
(2) $\mathcal{X}$ is a quotient stack if and only if there exists a variety X with the action of $\operatorname{GL}(n)$ such that

$$\mathcal{X} \cong X/\operatorname{GL}(n).$$

Relation between the motivic classes

In special cases, we have the following computations.

Lemma (Ekedahl, 2009)

In $\mathrm{Mot}(k)$ *we have the following results.*
1) $[GL(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$,
2) For a stack $\mathcal{X}$, *we have* $[\mathcal{X}/GL(n)] = [\mathcal{X}]/[GL(n)]$,
3) $[*/GL(n)] = 1/\left((\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})\right)$.

With the above lemma, we find a relation between the motivic classes.

Theorem (Ekedahl, 2009 and L, 2025)

There is a natural ring isomorphism :

$$\mathrm{Mot}_{var}(k)[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1} | i > 0] \cong \mathrm{Mot}(k).$$

Completion of the motivic classes

- Let $\mathrm{F}^m \operatorname{Mot}(k)$ be the subgroup generated by the classes of stacks of dimension $\leq -m$. This is a ring filtration and we define the completed ring $\overline{\operatorname{Mot}}(k)$ as the completion of $\operatorname{Mot}(k)$ with respect to this filtration.

The case of SL_n-torsor

Let X be a smooth geometrically connected projective curve of genus g over k. Let $\mathcal{B}un_{\mathrm{SL}_n}$ be the moduli stack parametrizing isomorphism classes of SL_n-torsor over X. The motivic classes may be computed as:

$$[\mathcal{B}un_{\mathrm{SL}_n}] = \mathbb{L}^{(n^2-1)(g-1)} \prod_{i=2}^{n} Z(X, \mathbb{L}^{-i}).$$

Motivic zeta function and plethystic exponents

The following theorem generalizes the motivic zeta function.

Theorem (Fedorov-Soibelman-Soibelman, 2018 and L, 2025)

One can uniquely extend the assignment X to Z_X to a continuous homomorphism of topological groups

$$Z : \overline{\mathrm{Mot}}(k) \to (1 + z\overline{\mathrm{Mot}}(k)[[z]])^{\times}$$

such that for any $A \in \overline{\mathrm{Mot}}(k)$ and any $n \in \mathbb{Z}$ we have $Z_{\mathbb{L}^n A}(z) = Z_A(\mathbb{L}^n z)$. More precisely, any class $A \in \overline{\mathrm{Mot}}(k)$ can be written as the limit of a sequence $([Y_i] - [Z_i])/\mathbb{L}^{n_i}$, where Y_i and Z_i are varieties. Thus we define

$$Z_A(z) = \lim_{i\to\infty} \frac{Z_{Y_i}(\mathbb{L}^{-n_i} z)}{Z_{Z_i}(\mathbb{L}^{-n_i} z)}.$$

Plethystic exponents

Definition

Now we can consider the ring of formal power series in two variables $\overline{\mathrm{Mot}}(k)[[z,w]]$. Let $\overline{\mathrm{Mot}}(k)[[z,w]]^+$ denote the ideal of power series with vanishing constant term and let $(1+\overline{\mathrm{Mot}}(k)[[z,w]]^+)^\times$ be the multiplicative group of series with constant term equal 1. We can define the plethystic exponent $\mathrm{Exp} : \overline{\mathrm{Mot}}(k)[[z,w]]^+ \to (1+\overline{\mathrm{Mot}}(k)[[z,w]]^+)^\times$ by

$$\mathrm{Exp}\left(\sum_{r,d} A_{r,d} w^r z^d\right) = \prod_{r,d} Z_{A_{r,d}}(w^r z^d).$$

Exp is an isomorphism of abelian groups, so we can define the plethystic logarithm Log as its inverse.

Stacks of Higgs bundles

Fix a smooth projective geometrically connected curve X over k.

Definition

A Higgs bundle on X is a pair (E, Φ) where E is a vector bundle on X and $\Phi : E \to E \otimes \Omega_X$ is an $\mathcal{O}_X$-linear morphism from E to E "twisted" by the sheaf of differential 1-forms Ω_X. The rank of the pair (E, Φ) is the rank of E, similarly the degree is the degree of E.

Definition

We denote by $\mathcal{H}iggs_{r,d}$ the moduli stack of rank r degree d Higgs bundles on X. This is an Artin stack locally of finite type over k.

Note that the motivic classes of $\mathcal{B}un_{r,d}$ converge, but the motivic classes of $\mathcal{H}iggs_{r,d}$ diverge, that is, have infinite volumes.

Moduli stacks of semistable Higgs bundles

Definition

The Higgs bundle (E, Φ) is called semistable if for any subbundle $F \subset E$ preserved by Φ,

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{\operatorname{rk} E}.$$

- This is an open condition compatible with field extensions, so we use $\mathcal{H}iggs^{ss}_{r,d} \subset \mathcal{H}iggs_{r,d}$ to denote the open substack of semistable Higgs bundles.
- $\mathcal{H}iggs^{ss}_{r,d}$ is a stack of finite type, thus the corresponding motivic classes converge.

Zeta functions and partitions

We give the definition of the motivic L-function, which is a polynomial of degree $2g$, where $g := g(X)$ is the genus of X:

$$L_X(z) := \zeta_X(z)(1-z)(1-\mathbb{L}z) \in \overline{\mathrm{Mot}}(k)[z].$$

Definition

We define "normalized" zeta-function by $\tilde{Z}_X(z) := z^{1-g} Z_X(z)$ and the "regularized" zeta-function by setting

$$Z_X^*(\mathbb{L}^{-u}z^v) = \begin{cases} Z_X^*(\mathbb{L}^{-u}z^v) \text{ if } v > 0 \text{ or } u > 1, \\ \dfrac{L_X(\mathbb{L}^{-1})}{1-\mathbb{L}^{-1}} \text{ if } (u,v) = (1,0), \\ \dfrac{L_X(1)}{1-\mathbb{L}} \text{ if } (u,v) = (0,0). \end{cases}$$

Explicit formulas I

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ be a partition. This can also be written as $\lambda = 1^{r_1} 2^{r_2} \ldots t^{r_t}$, where r_i denotes the number of occurrences of i among λ_j, $1 \leq j \leq l$. The Young diagram of λ is the set of the points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq i \leq \lambda_j$. For a box $s \in \lambda$, the arm $a(s)$ (resp. leg $l(s)$) is the number of boxes lying strictly to he right of (resp. strictly above) s.
For a partition λ, set

$$J_\lambda^{mot}(z) = \prod_{s \in \lambda} Z_X^*(\mathbb{L}^{-1-l(s)} z^{a(s)}) \in \overline{\mathrm{Mot}}(k)[[z]],$$

where the product is over all boxes of the Young diagram corresponding to the partition. In particular for the empty Young diagram λ we have $J_\lambda^{mot}(z) = 1$.

Explicit formulas II

Set

$$L^{mot}(z_n, \ldots, z_1) = \frac{1}{\prod_{i<j} \tilde{Z}_X\left(\frac{z_i}{z_j}\right)} \times$$

$$\times \sum_{\sigma \in S_n} \sigma \left\{ \prod_{i<j} \tilde{Z}_X\left(\frac{z_i}{z_j}\right) \frac{1}{\prod_{i<n}\left(1 - \mathbb{L}\frac{z_{i+1}}{z_i}\right)} \cdot \frac{1}{1 - z_1} \right\}.$$

Residue

Assume that $R(z)$ is a rational function with coefficients in $\overline{\mathrm{Mot}}(k)$, that is, an element of the total ring of fractions of $\overline{\mathrm{Mot}}[z]$. We say that it has at most first order pole at $x \in \overline{\mathrm{Mot}}$ if it can be written as

$$\frac{P(z)}{(x-z)Q(z)},$$

where $Q(x)$ is not a zero divisor in $\overline{\mathrm{Mot}}(k)$. In this case we can define

$$res_{z=x}R(z)dz = \frac{P(x)}{Q(x)}.$$

On the other hand, if $Q(x)$ is invertible, we can expand $R(z)$ in powers of z. The residue of the corresponding series may or may not exist, but if it exists, it is equal to the residue of the rational function.

Explicit formulas III

For a partition $\lambda = 1^{r_1}2^{r_2}\dots t^{r_t}$ such that $\sum_i r_i = n$, set $r_{<i} = \sum_{k<i} r_k$ and denote by res_λ the iterated residue along

$$\begin{array}{cccc} \frac{z_n}{z_{n-1}} = \mathbb{L}^{-1}, & \frac{z_{n-1}}{z_{n-2}} = \mathbb{L}^{-1}, & \dots, & \frac{z_{2+r_{<t}}}{z_{1+r_{<t}}} = \mathbb{L}^{-1}, \\ \vdots & \vdots & & \vdots \\ \frac{z_{r_1}}{z_{r_1-1}} = \mathbb{L}^{-1}, & \frac{z_{r_1-1}}{z_{r_1-2}} = \mathbb{L}^{-1}, & \dots, & \frac{z_2}{z_1} = \mathbb{L}^{-1}. \end{array}$$

Set

$$\tilde{H}^{mot}_\lambda(z_{1+r_{<t}}, \dots, z_{1+r_{<i}}, \dots, z_1) := \mathrm{res}_\lambda \left[L^{mot}(z_n, \dots, z_1) \prod_{\substack{j=1 \\ j \notin \{r_{<i}\}}}^{n} \frac{dz_j}{z_j} \right]$$

and

$$H^{mot}_\lambda(z) := \tilde{H}^{mot}_\lambda(z^t \mathbb{L}^{-r_{<t}}, \dots, z^i \mathbb{L}^{-r_{<i}}, \dots, z).$$

Formulas for semistable Higgs bundles

Theorem (Fedorov-Soibelman-Soibelman, 2018 and L, 2025)

For sufficiently large e, we have $[\mathcal{H}iggs^{ss}_{r,d}] = H_{r,d+er}$*, where* $H_{r,d}$ *is defined via the generating function*

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H_{r,d} w^r z^d = \mathrm{Exp}\left(\sum_{d/r=\tau} B_{r,d} w^r z^d \right),$$

where τ *is any rational number. We define* $B_{r,d}$ *by*

$$\sum_{\substack{r,d\in\mathbb{Z}_{\geq 0} \\ (r,d)\neq(0,0)}} B_{r,d} w^r z^d = \mathbb{L}\,\mathrm{Log}\left(\sum_{\lambda} \mathbb{L}^{(g-1)\langle\lambda,\lambda\rangle} J^{mot}_{\lambda}(z) H^{mot}_{\lambda}(z) w^{|\lambda|} \right).$$

The sum is over all partitions and $\langle\lambda,\lambda\rangle = \sum_i (\lambda'_i)^2$.

Bundles with connections

Fix a smooth geometrically connected projective curve X over k.

Definition

A bundle with connection on X is a pair (E, ∇) where E is a vector bundle on X and $\nabla : E \to E \otimes \Omega_X$ is a k-linear morphism of sheaves satisfying Leibniz rule, i.e. for any open subset U of X, any $f \in H^0(U, \mathcal{O}_X)$ and any $s \in H^0(U, E)$ we have

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

For the characteristic 0 case, we have

Theorem (Fedorov-Soibelman-Soibelman, 2018)

$$[\mathcal{C}onn_r(X)] = [\mathcal{H}iggs^{ss}_{r,0}(X)].$$

Moduli stacks of bundles with connections

Compared to the characteristic 0 case , where every vector bundle with a connection has degree zero, we have

Lemma (Biswas, 2005)

If k is a field and E is a vector bundle over X admitting a connection, then the degree of each indecomposable component of E is a multiple of the characteristic of k.

For this Lemma, k is not necessarily algebraically closed. If k is algebraically closed, the converse is also true.

Definition

We use $\mathcal{C}onn_{r,pd}$ to denote the moduli stack of rank r degree pd vector bundles with connections on X. In finite characteristic, the vector bundles with connections are not automatically semistable, so we restrict to the substack of semistable bundles with connections with the notation $\mathcal{C}onn^{ss}_{r,pd}$.

Formula for semistable bundles with connections

Conjecture (L, 2025)

For $char(k) = p$ and sufficiently large e, we have $[\mathcal{C}onn^{ss}_{r,pd}] = C_{r,p(d+er)}$, where $C_{r,pd}$ is defined via the generating function

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} C_{r,pd} w^r z^{pd} = \operatorname{Exp}\left(\sum_{d/r=\tau} B_{r,pd} w^r z^{pd} \right),$$

where τ is any rational number.

Thank you!